

Macroscopic equations for the conservation of heat (or the mass of a diffusing impurity) in a continuous medium containing distributed particles of a dispersed phase are formulated neglecting the effect of random fluctuations of the medium and particles by the transfer process. The problem of convective heat conduction or diffusion near an isolated particle is also formulated, the solution of which permits calculation of all the parameters entering into the indicated equations. This problem has been solved in the particular case of small Peclet numbers, which characterize heat and mass exchange in the vicinity of a single particle.

1. In many real situations the spatial scale L of the temperature or impurity concentration fields in a dispersed system significantly exceeds the system's internal structural scale l (for example, the average distance between adjacent particles of the dispersed phase), and only the global "macroscopic" characteristics of heat or mass transfer over distances comparable to L and significantly exceeding the indicated structural scale l are of fundamental interest. It is convenient in such cases to treat a dispersed system as the superposition of several coexisting continua [1], each of which is characterized by separate values of the average velocity, temperature, and so on; the variation of the latter quantities is described by the corresponding averaged conservation equations. The immediate problem evidently consists of a rigorous derivation of these equations.

The conservation equations of mass, momentum, and angular momentum of continua imitating the phases of a monodispersed suspension were derived in [2]. We apply below the methods developed in [2] to the derivation of the heat conservation equations in these continua. These equations will describe the heat transfer in a composite granular material in the particular case in which both phases of the dispersed medium are motionless. It is clear that all the results obtained for the description of heat transfer are also applicable to the analysis of the process of mass transport of an impurity diffusing in a dispersed medium.

Let $\mathbf{C}(t, \mathbf{r})$ and $T(t, \mathbf{r})$ be the functions describing the velocity and temperature fields in a dispersed system, and let $\theta(t, \mathbf{r})$ be a function equal to unity at points located in the system's continuous phase and to zero at points located inside the particles. Let us define the average velocities $\mathbf{v}(t, \mathbf{r})$, $\mathbf{w}(t, \mathbf{r})$ and temperatures $\tau_0(t, \mathbf{r})$, $\tau_1(t, \mathbf{r})$ of the continuous and dispersed phases in the following fashion, respectively:

$$\begin{aligned} \left. \begin{array}{l} \mathbf{v}(t, \mathbf{r}) \\ \tau_0(t, \mathbf{r}) \end{array} \right\} &= \frac{1}{\varepsilon(t, \mathbf{r}) b} \int_b \theta(t, \mathbf{r} + \mathbf{r}') \left. \begin{array}{l} \mathbf{C}(t, \mathbf{r} + \mathbf{r}') \\ T(t, \mathbf{r} + \mathbf{r}') \end{array} \right\} d\mathbf{r}' \\ \left. \begin{array}{l} \mathbf{w}(t, \mathbf{r}) \\ \tau_1(t, \mathbf{r}) \end{array} \right\} &= \frac{1}{\rho(t, \mathbf{r}) b} \int_b [1 - \theta(t, \mathbf{r} + \mathbf{r}')] \left. \begin{array}{l} \mathbf{C}(t, \mathbf{r} + \mathbf{r}') \\ T(t, \mathbf{r} + \mathbf{r}') \end{array} \right\} d\mathbf{r}' \end{aligned} \quad (1.1)$$

Here the integration is carried out over a small physical volume b , of the medium containing a sufficiently large number of particles and whose center of gravity is located at the point \mathbf{r} , and $\rho(t, \mathbf{r})$ and $\varepsilon(t, \mathbf{r})$ are the volume concentration of the dispersed phase and the porosity ("porousness") of the system, which can be determined as was done in [2] from

Translated from *Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki*, No. 4, pp. 79-87, July-August, 1974. Original article submitted January 4, 1974.

©1976 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for \$15.00.

$$\begin{aligned}\varepsilon(t, \mathbf{r}) &= \frac{1}{b} \int_b \theta(t, \mathbf{r} + \mathbf{r}') d\mathbf{r}' \\ \rho(t, \mathbf{r}) &= 4/3 \pi a^3 n(t, \mathbf{r}) = 1 - \varepsilon(t, \mathbf{r})\end{aligned}\quad (1.2)$$

where $n(t, \mathbf{r})$ is the countable (numerical) concentration of particles and a is the radius of a particle.

One can replace the volume average $b \sim l_b^3$ by an average over an arbitrarily oriented surface $s \sim l_b^2$, which plays the role of a small physical surface.

Let us further discuss the conservation of heat in a volume $B \gg b$ bounded by the surface S . The volume and surface of all the particle inside S are denoted by B_* and S_* . We introduce the symbols b_* and s_* for the particles inside the surface s which bounds the small physical volume b .

The local heat production per unit volume is described with the help of the function $H(t, \mathbf{r})$ which is defined at points occupied both by the continuous and the dispersed phases. The local equations of convective heat conduction are of the form

$$c_i (\partial / \partial t + \mathbf{C}\nabla) T = -\nabla \mathbf{Q}^{(i)} + H, \quad \mathbf{Q}^{(i)} = -\lambda_i \nabla T \quad (i = 0, 1) \quad (1.3)$$

Here \mathbf{Q} denotes the flux of heat generated by molecular heat conductivity and c and λ denote the specific heat capacity and the coefficient of molecular heat conductivity. The index values $i = 0$ and $i = 1$ in Eqs. (1.3) denote here and below the continuous and dispersed phases, respectively. The function $H(t, \mathbf{r})$ may represent both external heat sources, which are practically independent of the temperature field (for example, heat generated by the viscous disipation of flow energy) and sources whose strength is principally determined by this field (for example, heat generated by chemical reactions). In the latter case Eqs. (1.3) may be nonlinear.

The balance equations of the amount of heat in a dispersed system and in its continuous phase inside B are of the form

$$\begin{aligned}\frac{\partial}{\partial t} \int_B cT d\mathbf{r} + \int_S (cTC + \mathbf{Q}) \mathbf{n} d\mathbf{r} - \int_B H d\mathbf{r} &= 0 \\ \frac{\partial}{\partial t} \int_B cT\theta d\mathbf{r} + \int_S (cTC + \mathbf{Q}) \theta \mathbf{n} d\mathbf{r} + \int_{S_*} \mathbf{Q} \mathbf{n} d\mathbf{r} - \int_B H\theta d\mathbf{r} &= 0\end{aligned}\quad (1.4)$$

where \mathbf{n} denotes the unit vector of the outer normal of the volumes occupied by the system or the continuous phase. The integrals over B and S in the second relation of (1.4) can be replaced by integrals over $B-B_*$ and over the part of the surface S which passes through the continuous phase, respectively.

It is not difficult to show, as was done in [2], that to an accuracy to terms of the order of $N^{-1/3}$, where N is the number of particle in B , the equations are of the form

$$\int_S \mathbf{Q} \theta \mathbf{n} d\mathbf{r} + \int_{S_*} \mathbf{Q} \mathbf{n} d\mathbf{r} = \int_{S+S_*} \mathbf{Q} \mathbf{n} d\mathbf{r} = \int_{S+S_*} \mathbf{Q} \mathbf{n} d\mathbf{r} \quad (1.5)$$

where S_* is the surface of all particles lying entirely in B (i.e., not intersecting S). In addition, in accordance with the method in [2], we represent B and S as the superposition of small physical volumes b and surfaces s , and we carry out the integration in (1.4) and (1.5) in two stages - first we integrate over the individual b and s , and then we add up the results. Treating such expressions as integral sums and neglecting the resulting terms which are quadratic in the random deviation of local quantities from their average values defined in (1.1) and (1.2), we get

$$\begin{aligned}\frac{\partial}{\partial t} \int_B (c_0 \varepsilon \tau_0 + c_1 \rho \tau_1) d\mathbf{r} + \int_S (c_0 \varepsilon \tau_0 \mathbf{v} + c_1 \rho \tau_1 \mathbf{w}) \mathbf{n} d\mathbf{r} + \int_S \mathbf{q} \mathbf{n} d\mathbf{r} - \int_B (\varepsilon h_0 + \rho h_1) d\mathbf{r} &= 0 \\ \frac{\partial}{\partial t} \int_B c_0 \varepsilon \tau_0 d\mathbf{r} + \int_S c_0 \varepsilon \tau_0 \mathbf{v} \mathbf{n} d\mathbf{r} + \int_S \mathbf{q} \mathbf{n} d\mathbf{r} + \int_B (\psi - \varepsilon h_0) d\mathbf{r} &= 0\end{aligned}\quad (1.6)$$

Here the quantities

$$\begin{aligned}\psi(t, \mathbf{r}) &= -\frac{1}{b} \sum_j \int_{s_j} \mathbf{Q} \mathbf{n}' d\mathbf{r}, \quad \mathbf{q}(t, \mathbf{r}) = \frac{1}{b} \int_b \mathbf{Q}(t, \mathbf{r} + \mathbf{r}') d\mathbf{r}' \\ h_0(t, \mathbf{r}) &= \frac{1}{\varepsilon(t, \mathbf{r}) b} \int_b \theta(t, \mathbf{r} + \mathbf{r}') H(t, \mathbf{r} + \mathbf{r}') d\mathbf{r}' \\ h_1(t, \mathbf{r}) &= \frac{1}{\rho(t, \mathbf{r}) b} \int_b [1 - \theta(t, \mathbf{r} + \mathbf{r}')] H(t, \mathbf{r} + \mathbf{r}') d\mathbf{r}'\end{aligned}\quad (1.7)$$

are introduced.

The integration in the expression for $\psi(t, \mathbf{r})$ in (1.7) is carried out over the surface of the j -th particle, and the summation is over all the particles in b ; \mathbf{n}' denotes the unit vector of the outer normal of the particle. It is clear that

$$\psi(t, \mathbf{r}) = -n(t, \mathbf{r}) \oint \langle \mathbf{Q} \rangle \mathbf{n}' d\mathbf{r} \quad (1.8)$$

where the angular brackets denote an average over the ensemble of particles, and the integration is performed over the surface of a single particle.

From Eqs. (1.2), (1.3), and (1.7) we obtain the following expression for the average heat flux $\mathbf{q}(t, \mathbf{r})$ associated with molecular thermal conduction:

$$\mathbf{q}(t, \mathbf{r}) = -\frac{1}{b} \left(\lambda_0 \int_{b-b_*} \frac{\partial T}{\partial \mathbf{r}} \Big|_{t, \mathbf{r}+\mathbf{r}'} d\mathbf{r}' + \lambda_1 \int_{b_*} \frac{\partial T}{\partial \mathbf{r}} \Big|_{t, \mathbf{r}+\mathbf{r}'} d\mathbf{r}' \right) = -\lambda_0 \frac{1}{b} \int_b \frac{\partial T}{\partial \mathbf{r}} \Big|_{t, \mathbf{r}+\mathbf{r}'} d\mathbf{r}' - (\lambda_1 - \lambda_0) n(t, \mathbf{r}) \oint \langle T \rangle \mathbf{n}' d\mathbf{r} \quad (1.9)$$

Finally, the functions $h_0(t, \mathbf{r})$ and $h_1(t, \mathbf{r})$ which are determined by averaging the function $H(t, \mathbf{r})$, depends significantly on the specific form of this function. Below we will consider these quantities as known. In the general case they may depend on the average temperatures $\tau_0(t, \mathbf{r})$, and $\tau_1(t, \mathbf{r})$.

We have from the first relation of (1.6) the following differential equation for heat transfer in a dispersed medium:

$$\partial/\partial t (c_0 \varepsilon \tau_0 + c_1 \rho \tau_1) + \nabla (c_0 \varepsilon \tau_0 \mathbf{v} + c_1 \rho \tau_1 \mathbf{w}) = -\nabla \mathbf{q} + \varepsilon h_0 + \rho h_1 \quad (1.10)$$

A similar equation for the continuous phase follows from the second relation of (1.6), and the equation for the dispersed phase is the difference between (1.10) and the heat transfer equation in the continuous phase. Thus, we get

$$\begin{aligned} \partial/\partial t (c_0 \varepsilon \tau_0) + \nabla (c_0 \varepsilon \tau_0 \mathbf{v}) &= -\nabla \mathbf{q} - \psi + \varepsilon h_0 \\ \partial/\partial t (c_1 \rho \tau_1) + \nabla (c_1 \rho \tau_1 \mathbf{w}) &= \psi + \rho h_1 \end{aligned} \quad (1.11)$$

Using the conservation equations of the mass of the phases derived in [2], we also get from (1.10) and (1.11)

$$\begin{aligned} c_0 \varepsilon (\partial/\partial t + \mathbf{v} \nabla) \tau_0 + c_1 \rho (\partial/\partial t + \mathbf{w} \nabla) \tau_1 &= -\nabla \mathbf{q} + \varepsilon h_0 + \rho h_1 \\ c_0 \varepsilon (\partial/\partial t + \mathbf{v} \nabla) \tau_0 &= -\nabla \mathbf{q} - \psi + \varepsilon h_0 \\ c_1 \rho (\partial/\partial t + \mathbf{w} \nabla) \tau_1 &= \psi + \rho h_1 \end{aligned} \quad (1.12)$$

It is clear that the first equation of (1.12) is the sum of the other two.

Thus we arrive at a model according to which a dispersed medium is represented in the form of a superposition of two continua having not only different velocities but different temperatures; the transfer of heat in these continua is described by the last two equations of (1.12), and the equations determining their motion has been formulated in [2]. (We emphasize that we are talking only about a monodispersed system; in the general case of a system containing particles of various sizes the number of such continua may be larger.) Equations (1.12) contain the unknown quantities $\psi(t, \mathbf{r})$ and $\mathbf{q}(t, \mathbf{r})$ which according to (1.8) and (1.9) can be determined from the average temperatures and the heat flux at the surface of an individual particle. The problem of determining them as functions or functionals of the unknown variables in (1.12) and the physical parameters is analogous in a sense to the problem of deriving the rheological equations of the state of a suspension, which is encountered in setting up its continuum hydromechanics and is discussed in detail in [2, 3].

2. Following the general method [3], we formulate an averaged equation characterizing heat transfer in a continuous phase near an isolated (test) particle whose solution will permit calculation of the average temperature and heat flux at the surface of the test particle and, thereby, calculation of $\psi(t, \mathbf{r})$ and $\mathbf{q}(t, \mathbf{r})$ from (1.8) and (1.9) and closure of the system of equations (1.12). Thus it is necessary to introduce the concepts of an ensemble of possible spatial arrangements (configurations) of the particles of the dispersed phase and of the procedures for averaging over the ensemble. Taking into account the fact that the properties of an ensemble of configurations are elucidated in detail in [3], and have also been discussed in [4, 5] with application to the problem of the effective heat conductivity of composite granular materials, we use below the mathematical apparatus of averaging from [3-5] without special stipulations.

We note here only the principal assumptions adopted below. In the first place, the arrangement of the particles is assumed to be random (chaotic). Secondly, the condition of "relaxation of correlations" is adopted, according to which the conditional averages, obtained by means of averaging over the particle distribution function on the condition that some specific point is occupied by the center of the test particle, tend to the appropriate unconditional averages upon removal from this point. Thirdly, it is assumed that the average temperature of the continuous phase at some point coincides approximately with the quantity derived by averaging the temperature only over such configurations for which the presence of one of the particles of this point would be possible. The degree of validity of these assumptions for materials with stationary phases and for moving suspensions, with boundaries imposed on them, and also the possible ways of generalizing the results derived on the basis of these assumptions are discussed in detail in [3].

We will first derive the equation describing heat transfer in a continuous phase "on the average", using for this purpose an unconditional average over the ensemble. Multiplying Eq. (1.3) by the function $\theta(t, \mathbf{r})$ and averaging, we obtain

$$c_0 \langle \theta (\partial / \partial t + \mathbf{C}\nabla) T \rangle = - \nabla \langle \theta \mathbf{Q} \rangle + \langle \mathbf{Q} \nabla \theta \rangle + \langle \theta H \rangle \quad (2.1)$$

Equations (2.1) contains the same physical information as does the analogous equation derived by averaging over the volume, i.e, the second equation of (1.12). The condition that these equations agree amounts to the content of the ergodic condition, which is similarly formulated in [3]. Averaging the indicated equations, we obtain the following relations linking the volume and ensemble averages:

$$\begin{aligned} \langle \theta (\partial / \partial t + \mathbf{C}\nabla) T \rangle &= - \varepsilon (\partial / \partial t + \mathbf{v}\nabla) \tau_0 - \nabla \langle \theta \mathbf{Q} \rangle + \langle \mathbf{Q} \nabla \theta \rangle = \lambda_0 \nabla \frac{1}{b} \int_b \frac{\partial T}{\partial \mathbf{r}} \Big|_{t, \mathbf{r}+\mathbf{r}'} d\mathbf{r}' \\ &+ (\lambda_1 - \lambda_0) \nabla \left(n \oint \langle T \rangle \mathbf{n}' dr \right) + n \oint \langle Q \rangle \mathbf{n}' dr \langle \theta T \rangle = \varepsilon \tau_0, \langle \theta H \rangle = \varepsilon h_0 \end{aligned} \quad (2.2)$$

Here the definitions (1.8) and (1.9) were used.

Using the explicit expression for $\theta(t, \mathbf{r})$,

$$\theta(t, \mathbf{r}) = \sum_{j=1}^N \frac{\mathbf{r} - \mathbf{r}^{(j)}}{|\mathbf{r} - \mathbf{r}^{(j)}|} \delta(|\mathbf{r} - \mathbf{r}^{(j)}| - a) = \sum_{j=1}^N \mathbf{n}' \delta(|\mathbf{r} - \mathbf{r}^{(j)}| - a) \quad (2.3)$$

where $\mathbf{r}^{(j)}$ are the radius vectors of the particles' centers, it is easy to derive an expression for $\langle \mathbf{Q} \nabla \theta \rangle$ at some point \mathbf{r} in the form of an integral over the possible positions of the center of a physical particle in such a way that the point \mathbf{r} lies on the particle's surface [6]. Resolving the integrand into a series according to powers of the components of the vector which connects the particle's center with \mathbf{r} , and integrating the series term by term we arrive at an expression for $\langle \mathbf{Q} \nabla \theta \rangle$ in the form of distributed thermal multipoles. Thus heat transfer in a continuous phase occupying the space between particles can be formally described as heat transfer in some fictitious medium filling all the space and containing distributed thermal sources, dipoles and so on.

We note that such a description of the effect of particles on heat transfer in a continuous phase is similar in concept to the description of the effect of particles on the motion of a liquid in the spaces between them which has been proposed in [3]. The multipole exposition in question was derived in another more phenomenological manner in [4, 5], where only the "dipole approximation" was assumed, however, in which the effect of thermal quadrupoles and multipoles of higher rank was neglected. The dipole approximation was also used in [7] in calculating the effective thermal conductivity of grainy materials with spheroidal inclusions. We also note that the notion of a fictitious medium surrounding the test particle was introduced earlier on phenomenological grounds. There is a review of the papers about this trend in [8].

The equation describing heat transfer near a test particle with center at the point $\mathbf{r} = 0$ is derived just as (2.1) was if one uses conditional averaging only for this configurations of all the remaining particles which are compatible with the fact that the center of the test particle is situated at a specific point. Denoting such averaging by angular brackets, with a prime and neglecting as before, terms which are quadratic with respect to the fluctuations of the quantities relative to their average values, we obtained

$$c_0 \langle \langle \theta \rangle' (\partial / \partial t + \mathbf{U}\nabla) \langle T \rangle' \rangle = - \nabla \langle \langle \theta \mathbf{Q} \rangle' \rangle + \langle \langle \mathbf{Q} \nabla \theta \rangle' \rangle + \langle \langle \theta H \rangle' \rangle \quad (2.4)$$

in the system of coordinates associated with the center of the test particle.

This equation is of the same type as has been discussed in [3], whereby one can consider the quantity $U = \langle \theta C \rangle'$ which has been investigated in detail in [3], as a known function of the coordinates and the time. Conditional averages of the form $\langle \varphi \rangle'$ in (2.4) depend on r and are transformed into quantities characteristic of a pure phase on the surface of the test particle (i.e., at $r = a$) and into quantities specific to the fictitious medium introduced when $r = 3a$ (compare with an analogous situation in [3]).

Determination of the specific dependence of the quantities $\langle \varphi \rangle'$ on r near the surface of a test particle is very complicated, but it is evidently sufficient for practical purposes to consider two approximate models. The first of these is valid for systems of moderate concentration up to $\rho \approx 0.2-0.3$. Within the framework of this model one can generally neglect the existence of the layer of abrupt change in parameters at the surface $r = a$, having simply set $\langle \varphi \rangle' = \langle \varphi \rangle$. As follows from the analysis in [3], this corresponds to the neglect of the effect of the nonoverlapping of particles of the dispersed phase. Within the framework of the second model the layer in question is approximated by a layer $a \leq r \leq 2a$ filled with a pure continuous phase. Such an approximation appears exact if one considers the problem in the approximation in which only a finite number of distributed thermal multipoles is taken into account, as was done, for example, in [4, 5, 7]. The phenomenological notion of such a layer was introduced in [9].

In the first case heat transfer outside the test particle is described actually by Eq. (2.1). In the second case Eq. (2.1) is valid only for $r > 2a$, and heat transfer in the layer $a \leq r \leq 2a$ is governed by the usual equation of convective heat conductivity (1.3) for a pure continuous phase. The continuity conditions of temperature and heat flux should obviously be fulfilled on the surface $r = 2a$.

Heat transfer inside the test particle is described by the equation

$$c_1(\partial / \partial t + \mathbf{W}\nabla) \langle T \rangle' = \lambda_1 \Delta \langle T \rangle' + \langle H \rangle', \quad \langle H \rangle' = h_1 \quad (2.5)$$

where W is the velocity produced by a possible rotation of the test particle. Conditions of temperature and heat flux continuity are also imposed at the surface $r = a$.

The problem formulated will be closed if we find an expression for the quantity $-\nabla \langle \theta Q \rangle + \langle Q \nabla \theta \rangle$, which appears in the equation of heat transfer in a fictitious medium in the form of a function on functional of the unknown temperature $\langle T \rangle'(t, r)$ and $\tau_0(t, r)$. Using the problem's linearity and concepts completely analogous to those in [3], we present

$$-\nabla \langle \theta Q \rangle + \langle Q \nabla \theta \rangle = \int [\sigma(\omega) T_0^{(\omega)}(\omega, r) + \lambda_a(\omega) \Delta T_0^{(\omega)}(\omega, r)] e^{i\omega t} d\omega \quad (2.6)$$

where $T_0^{(\omega)}(\omega, r)$ is the Fourier transform of the quantity $\langle T \rangle'(t, r)$. The coefficients $\lambda_a(\omega)$ and $\sigma(\omega)$ characterize respectively the "apparent" thermal conductivity of the continuous phase and heat exchange between the phases [4, 5]. They should be determined a posteriori from the second condition of (2.2).

The relation between $\langle T \rangle'(t, r)$ and $\tau_0(t, r)$ is given by the expression

$$\lim_{r \rightarrow \infty} \langle T \rangle'(t, r) = \tau_0(t, r) \quad (2.7)$$

which definitely closes the problem of thermal conductivity near a test particle.

Formulation of the boundary condition (2.7) requires some explanation. The functional $\langle T \rangle'(t, r)$ differs significantly from the average temperature $\tau_0(t, r)$ only at distances $r - a \sim l \sim a$, from the test particle's surface and practically coincides with it when $r - a \gg l$, whereby in the region $r - a < l$ the quantity $\tau_0(t, r)$ can be considered as approximately independent of r . Taking into account the adopted assumption $L \gg l$ and using the main idea of the method of coalesced asymptotic expansions, the quality of the quantities $\langle T \rangle'(t, r)$ and $\tau_0(t, r)$ in the region $r - a \gg l$ can be replaced by the asymptotic relation (2.7), which is imposed when $r - a \rightarrow \infty$.

It follows from the form of Eq. (1.12) that for a steady-state process of heat transfer in a stationary granular material it is convenient to consider the single equation

$$-\nabla \mathbf{q} + h = 0, \quad h = \varepsilon h_0 + \rho h_1 \quad (2.8)$$

It is useful in this case to introduce the material's average temperature

$$\tau = \varepsilon \tau_0 + \rho \tau_1 \quad (2.9)$$

and its effective thermal-conductivity coefficient, defined as the coefficient of proportionality between the average heat flux q and the gradient of the average temperature $\nabla \tau$. Denoting this coefficient by λ , we write

$$-\nabla \langle \theta Q \rangle + \nabla \langle Q \nabla \theta \rangle = \lambda \Delta \langle T \rangle \quad (2.10)$$

in place of (2.6) in the case under consideration.

The dependence of λ on the material's parameters is also determined from the second relation of (2.2) after solution of the problem of heat conduction near a test particle. This very problem is discussed below for illustration; its solution also permits refining the expression for the effective thermal conductivity of granular materials derived in [4, 5].

3. On the basis of the results in Section 2 the problem of steady-state heat conduction near a test particle of a moderately concentrated material with stationary phases can be written in the form

$$\begin{aligned} \Delta T &= \Delta T_1 = 0, \quad T = \tau \quad (r \rightarrow \infty) \\ T &= T_1, \quad \lambda \mathbf{n} \nabla T = \lambda_1 \mathbf{n} \nabla T_1 \quad (r = a) \end{aligned} \quad (3.1)$$

The angular brackets are omitted for simplicity in the symbols for the independent variables; T and T_1 in (3.1) represent the average temperature outside and inside the test particle, respectively, and the temperature $\tau(\mathbf{r})$ is defined in (2.9) and is treated here as a known function. We note that the same problem is encountered for a test particle of a moving suspension if the Peclet number, which characterizes heat exchange with the test particle, is small and the heat transfer process is close to steady-state

We will represent $\tau(\mathbf{r})$ in the form of an expansion in the spherical functions $Y_n(\vartheta, \varphi)$ where ϑ and φ are the angular coordinates of the spherical coordinate system.

$$\tau(\mathbf{r}) = \sum_n A_n r^n Y_n(\vartheta, \varphi) \quad (3.2)$$

and the A_n are considered known coefficients. Solution of the problem (3.1) and (3.2) is the form

$$T = \sum_n (\alpha_n r^n + \gamma_n r^{-(n+1)}) Y_n, \quad T_1 = \sum_n \alpha_n^{(1)} r^n Y_n \quad (3.3)$$

The coefficients α_n , γ_n and $\alpha_n^{(1)}$ are determined from the equations

$$\begin{aligned} \alpha_n &= A_n, \quad \alpha_n + \gamma_n a^{-(2n+1)} = \alpha_n^{(1)} \\ n \alpha_n - (n+1) \gamma_n a^{-(2n+1)} &= n \alpha_n^{(1)} \end{aligned} \quad (3.4)$$

One can see that only the terms of (3.3) with $n = 1$ make a contribution to the integrals over the surface of the test particle which figure in (2.2). Therefore it is sufficient to consider only the particular solution of the problem which corresponds to the function $\tau(\mathbf{r}) = \mathbf{E} \mathbf{r}$, where \mathbf{E} is the gradient of the material's average temperature. Such a solution is of the form

$$T = \left[1 - \frac{\lambda_1 - \lambda}{\lambda_1 + 2\lambda} \left(\frac{a}{r} \right)^3 \right] \mathbf{E} \mathbf{r}, \quad T_1 = \frac{3\lambda}{\lambda_1 + 2\lambda} \mathbf{E} \mathbf{r} \quad (3.5)$$

where λ is the material's effective thermal conductivity introduced in (2.10). Using (1.9) and (3.5), we obtain

$$\mathbf{q} = -\lambda \mathbf{E} = - \left[\lambda_0 + \frac{3\lambda(\lambda_1 - \lambda_0)}{\lambda_1 + 2\lambda} \rho \right] \mathbf{E} \quad (3.6)$$

from the second relation of (2.2).

From this there follows an equation for λ whose solution is

$$\begin{aligned} \lambda &= \beta \lambda_0, \quad \varkappa = \lambda_1 / \lambda_0 \\ \beta &= 1/4 \{ 2 - 3\rho - \varkappa(1-3\rho) + [(2-3\rho - \varkappa(1-3\rho))^2 + 8\varkappa]^{1/2} \} \end{aligned} \quad (3.7)$$

It is not difficult to also derive asymptotic formulas valid when $\varkappa \rightarrow 0$ (for example, for porous bodies) and when $\varkappa \rightarrow \infty$ (for example, for propellants having low thermal conductivity and containing metallic inclusions). In the second case, the quantity $\beta = \lambda/\lambda_0$ remains finite if $\rho < 1/3$, and it becomes infinite in proportion to $\varkappa = \lambda_1/\lambda_0$ if $\rho > 1/3$. The latter situation is caused by the fact that when ρ is large the model under consideration, valid only for moderately concentrated materials, ceases to be applicable [4, 5]. The dependence of β on \varkappa from (3.7) is shown in Fig. 1 as the dashed lines for various values of ρ .

In place of (3.1) we have

$$\begin{aligned} \Delta T &= \Delta T_* = \Delta T_1 = 0 \quad T = \tau \quad (r \rightarrow \infty) \quad T = T_*, \quad \lambda \mathbf{n} \nabla T \\ &= \lambda_0 \mathbf{n} \nabla T_* \quad (r = 2a) \quad T_* = T_1, \quad \lambda_0 \mathbf{n} \nabla T_* = \lambda_1 \mathbf{n} \nabla T_1 \quad (r = a) \end{aligned} \quad (3.8)$$

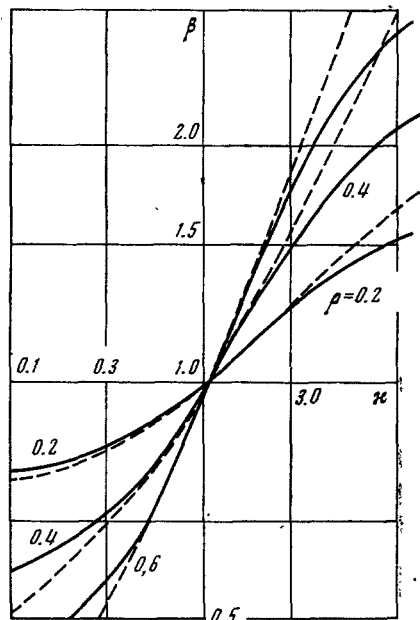


Fig. 1

for materials with a high concentration of the dispersed phase.

The solution of this problem with $\tau(\mathbf{r}) = \mathbf{E}r$ was obtained in [4,5]. In place of (3.6) we obtain in the present case

$$\mathbf{q} = -\lambda \mathbf{E} = -\left[\lambda_0 + \frac{36\lambda(\lambda_1 - \lambda_0)\lambda_0}{\lambda(7\lambda_1 + 17\lambda_0) + \lambda_0(5\lambda_1 + 7\lambda_0)} \rho \right] \mathbf{E} \quad (3.9)$$

for the quantity q .

The solution of the equation corresponding to (3.9) for $\beta = \lambda/\lambda_0$ is

$$\beta = (7\kappa + 17)^{-1} \{ \kappa(1 + 18\rho) + 5 - 18\rho + [(\kappa(1 + 18\rho) + 5 - 18\rho)^2 + (7\kappa + 17)(5\kappa + 7)]^{1/2} \} \quad (3.10)$$

The quantity β from (3.10) remains finite for any κ and ρ ; its dependence on κ , and ρ is illustrated by the solid lines in Fig. 1.

For small ρ and values of κ close to unity Eqs. (3.7) and (3.10) give practically identical results, coinciding with the well-known Maxwell equation when $\rho \rightarrow 0$.

We note that the dependences of the effective thermal conductivity on ρ and κ , which follow from (3.7) and (3.10) differ somewhat from the analogous dependences in [4, 5], which were derived within the framework of the dipole approximation, and can be considered to be refinements of the latter.

Equations (3.6) or (3.9) permit closing Eq. (2.8), which describes steady-state thermal conduction in a granular material. In the general case in which, first of all, the process is essentially non-steady-state and, secondly, the material's phases are moving, whereby the Peclet number for an individual particle is not small, it is necessary to solve the significantly more complex problem of convective heat conduction in the vicinity of a test particle for closure of the system (1.12) of macroscopic heat transfer equation. We emphasize that it is more convenient in this case to consider the apparent thermal conductivity λ_a of a fictitious medium [i.e., use (2.6) but not (2.10)], but the effective thermal conductivity of the material "as a whole" is generally devoid of physical meaning.

We note that in view of the analogy of the mathematical formulations of corresponding problems [4, 5], Eqs. (3.7) and (3.10) can be used not only for calculating the effective thermal conductivity or diffusion coefficient of an impurity in a dispersed medium but also for determining the dielectric or magnetic constants and electrical conductivity of dispersed materials.

LITERATURE CITED

1. Kh. A. Rakhmatulin, "Fundamentals of the gasdynamics of interpenetrating motions of compressible media," *Prikl. Matem. Mekhan.*, **20**, No. 2 (1956).
2. Yu. A. Buevich and V. G. Markov, "Continuum mechanics of monodispersed suspension. Integral and differential conservation laws," *Prikl. Matem. Mekhan.*, **37**, No. 5 (1973).
3. Yu. A. Buevich and V. G. Markov, "Continuum mechanics of monodispersed suspensions. Rheological equations of state for suspensions of moderate concentration," *Prikl. Matem. Mekhan.*, **37**, No. 6 (1973).
4. Yu. A. Buevich, "The effective thermal conductivity of granular materials," *Zh. Prikl. Mekhan. Tekh. Fiz.*, No. 4 (1973).
5. Yu. A. Buevich (Buyevich), "On the thermal conductivity of granular materials," *Chem. Eng. Sci.*, **29**, No. 1 (1974).
6. T. S. Lundgren, "Slow flow through stationary random beds and suspensions of spheres," *J. Fluid Mech.*, **51**, Pt. 2 (1972).
7. I. N. Shchelchkova, "Effective thermal conductivity of a heterogeneous medium with ellipsoidal particles," *Zh. Prikl. Mekhan. Tekh. Fiz.*, No. 1 (1974).
8. Z. Hashin, "Assesment of the self-consistent scheme approximation: conductivity of particular composites," *J. Composite Materials*, **2**, No. 3 (1968).
9. E. H. Kerner, "The electrical conductivity of composite media," *Proc. Phys. Soc.*, **69**, No. 8 (1956).